

# MAXIMA OF A TRIANGULAR ARRAY OF MULTIVARIATE GAUSSIAN SEQUENCE

ENKELEJD HASHORVA, LIANG PENG, AND ZHICHAO WENG

**Abstract:** It is known that the normalized maxima of a sequence of independent and identically distributed bivariate normal random vectors with correlation coefficient  $\rho \in (-1, 1)$  is asymptotically independent, which may seriously underestimate extreme probabilities in practice. By letting  $\rho$  depend on the sample size and go to one with certain rate, Hüsler and Reiss (1989) showed that the normalized maxima can become asymptotically dependent. In this paper, we extend such a study to a triangular array of multivariate Gaussian sequence, which further generalizes the results in Hsing, Hüsler and Reiss (1996) and Hashorva and Weng (2013).

**Key Words:** Correlation coefficient; maxima; stationary Gaussian triangular array

**AMS Classification:** Primary 60G15; secondary 60G70

## 1. INTRODUCTION

Let  $(X_1^{(1)}, X_1^{(2)}), \dots, (X_n^{(1)}, X_n^{(2)})$  be independent and identically distributed bivariate normal random vectors with zero means, unit variances and correlation coefficient  $\rho \in [-1, 1]$ . Put

$$(1.1) \quad u_n(x) = x/a_n + b_n \quad \text{with} \quad a_n = \sqrt{2 \ln n} \quad \text{and} \quad b_n = \sqrt{2 \ln n} - \frac{\ln \ln n + \ln(4\pi)}{2\sqrt{2 \ln n}}.$$

When  $|\rho| < 1$ , it is known that for any  $x, y \in \mathbb{R}$

$$\Psi_\rho(u_n(x), u_n(y)) := \mathbb{P} \left( \max_{1 \leq i \leq n} X_i^{(1)} \leq u_n(x), \max_{1 \leq i \leq n} X_i^{(2)} \leq u_n(y) \right) \rightarrow e^{-e^{-x} - e^{-y}} \quad \text{as } n \rightarrow \infty,$$

where the limit becomes the joint distribution of two independent Gumbel random variables. In this case,  $X_1^{(1)}$  and  $X_1^{(2)}$  are called asymptotically independent. Although normal distributions have many good properties and receive much attention in risk management (see McNeil, Frey and Embrechts (2005) for some overviews), this asymptotic independence property does seriously underestimate certain extreme probabilities in practice. To overcome this drawback, Hüsler and Reiss (1989) proposed to let  $\rho = \rho(n)$  depend on the sample size  $n$  such that

$$(1.2) \quad (1 - \rho(n)) \ln n \rightarrow \lambda \in [0, \infty] \quad \text{as } n \rightarrow \infty,$$

and then showed that

$$(1.3) \quad \lim_{n \rightarrow \infty} \Psi_{\rho(n)}(u_n(x), u_n(y)) = e^{-\Phi(\sqrt{\lambda} + \frac{x-y}{2\sqrt{\lambda}})e^{-y} - \Phi(\sqrt{\lambda} + \frac{y-x}{2\sqrt{\lambda}})e^{-x}} =: H_\lambda(x, y)$$

for  $x, y \in \mathbb{R}$ , where  $\Phi$  denotes the standard normal distribution function. It is easy to see that the limit distribution  $H_\lambda$  (referred to as the Hüsler-Reiss distribution) is not a product distribution when  $\lambda \in (0, \infty)$ , i.e.,  $X_1^{(1)}$  and  $X_1^{(2)}$  are asymptotically dependent in this case. Using (1.2), Frick and Reiss (2013) extended the above limit to the maxima of normal copulas. Some other extensions of Hüsler and Reiss (1989) to more general triangular arrays have been made in the literature too as reviewed below.

Consider a triangular array of normal random variables  $X_{n,i}, i = 1, 2, \dots, n = 1, 2, \dots$  such that for each  $n$ ,  $\{X_{n,i}, i \geq 1\}$  is a stationary normal sequence with mean zero, variance one and covariance  $\rho_{n,j} = \mathbb{E}\{X_{n,1}X_{n,j+1}\}$ . Motivated by condition (1.2), by assuming that

$$(1.4) \quad (1 - \rho_{n,j}) \ln n \rightarrow \delta_j \in (0, \infty] \quad \text{for all } j \geq 1$$

as  $n \rightarrow \infty$ , and some other conditions on  $\rho_{n,j}$ , Hsing, Hüsler and Reiss (1996) showed that

$$(1.5) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left( \max_{1 \leq j \leq n} X_{n,j} \leq u_n(x) \right) = e^{-\theta e^{-x}}$$

holds for all  $x \in \mathbb{R}$ , where

$$\theta = \mathbb{P} \left( A/2 + \sqrt{\delta_k} W_k \leq \delta_k \quad \text{for all } k \geq 1 \quad \text{such that } \delta_k < \infty \right),$$

with  $A$  being a standard exponential random variable independent of  $W_k$  and  $\{W_k : \delta_k < \infty, k \geq 1\}$  being jointly normal with zero means and

$$\mathbb{E}\{W_i W_j\} = \frac{\delta_i + \delta_j - \delta_{|i-j|}}{2\sqrt{\delta_i \delta_j}}.$$

Here  $\theta$  is set to be 1 if all  $\delta_j$ 's are infinite. Recently French and Davis (2013) generalized this study to a Gaussian random field on a lattice.

Another extension of Hüsler and Reiss (1989) made by Hashorva and Weng (2013) is to study a triangular array of 2-dimensional stationary Gaussian sequence as follows.

Consider a triangular array of bivariate normal random vectors  $X_{n,j} = (X_{n,j}^{(1)}, X_{n,j}^{(2)}), j = 1, 2, \dots, n = 1, 2, \dots$  such that for each  $n$ ,  $\{X_{n,j}, j \geq 1\}$  is a Gaussian sequence with mean zero, variance one and covariance

$$\mathbb{E}\{X_{n,k}^{(i)} X_{n,l}^{(j)}\} = \rho_{ij}(|k-l|, n) \quad \text{for } i, j = 1, 2.$$

By assuming that

$$(1.6) \quad \lim_{n \rightarrow \infty} (1 - \rho_{12}(0, n)) \ln n = \lambda \in [0, \infty]$$

and

$$(1.7) \quad \sigma := \max_{1 \leq k < n, 1 \leq i, j \leq 2} |\rho_{ij}(k, n)| < 1, \quad \lim_{n \rightarrow \infty} \max_{l_n \leq k < n, 1 \leq i, j \leq 2} \rho_{ij}(k, n) \ln n = 0,$$

where  $l_n = [n^\alpha]$  for some  $\alpha \in (0, \frac{1-\sigma}{1+\sigma})$ , Hashorva and Weng (2003) proved that

$$(1.8) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left( \max_{1 \leq k \leq n} X_{n,k}^{(1)} \leq u_n(x), \max_{1 \leq k \leq n} X_{n,k}^{(2)} \leq u_n(y) \right) = H_\lambda(x, y)$$

for all  $x, y \in \mathbb{R}$ . Taking  $y = \infty$  in (1.8), we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \max_{1 \leq k \leq n} X_{n,k}^{(1)} \leq u_n(x) \right) = e^{-e^{-x}} \quad \text{for } x \in \mathbb{R},$$

which may contradict (1.5). Note that when (1.4) holds for  $X_{n,k}^{(1)}$  or/and  $X_{n,k}^{(2)}$ , we have  $\lim_{n \rightarrow \infty} \sigma = 1$  and  $S_{n1}$  does not converge to zero (see the bottom of page 323 in Hashorva and Weng (2013)). That is, convergence in (1.8) excludes the possibility that (1.4) holds for  $X_{n,k}^{(1)}$  and  $X_{n,k}^{(2)}$ . This motivates us to investigate the limit of  $\mathbb{P} \left( \max_{1 \leq k \leq n} X_{n,k}^{(1)} \leq u_n(x), \max_{1 \leq k \leq n} X_{n,k}^{(2)} \leq u_n(y) \right)$  when (1.6) holds and (1.4) holds for both  $X_{n,k}^{(1)}$  and  $X_{n,k}^{(2)}$ . Such a study will generalize the results in both Hsing, Hüsler and Reiss (1996) and Hashorva and Weng (2013).

Some other recent extensions of Hüsler and Reiss (1989) is to drop the Gaussian assumption. For example, Hashorva (2013) studied the maxima of some spherical processes; Hashova, Kabluchko and Wübker (2012) investigated the

maxima of  $\chi^2$ -random vectors; Manjunath, Frick and Reiss (2012) discussed the maxima in the setup of extremal discriminant analysis; Engelke, Kabluchko and Schlather (2014) analyzed the maxima for some type of conditional Gaussian models.

We organize this paper as follows. Section 2 derives the limit for the normalized componentwise maxima of a triangular array of  $d$ -dimensional normal random vectors when (1.4) holds for both marginals and dependence. All proofs are put in Section 3.

## 2. MAIN RESULTS

Throughout we consider a triangular array  $\mathbf{X}_{n,k} = (X_{n,k}^{(1)}, \dots, X_{n,k}^{(d)}), k = 1, 2, \dots, n = 1, 2, \dots$  such that for each  $n$ ,  $\{\mathbf{X}_{n,k}, k \geq 1\}$  is a  $d$ -dimensional stationary Gaussian sequence with mean zero, variance one and correlations given by  $\mathbb{E}\{X_{n,k}^{(i)} X_{n,l}^{(j)}\} = \rho_{ij}(|k-l|, n)$  for  $k, l = 1, 2, \dots$  and  $i, j = 1, 2, \dots, d$ .

Hereafter  $A$  stands for a unit exponential random variable being independent of all other random elements involved and  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ .

**Theorem 2.1.** *Let  $\{\mathbf{X}_{n,k}, k, n \geq 1\}$  be a  $d$ -dimensional stationary Gaussian triangular array satisfying*

$$(2.1) \quad \begin{cases} \lim_{n \rightarrow \infty} (1 - \rho_{ij}(k, n)) \ln n = \delta_{ij}(k) \in (0, \infty] & \text{for } i, j = 1, \dots, d; k = 1, 2, \dots \\ \lim_{n \rightarrow \infty} (1 - \rho_{ij}(0, n)) \ln n = \delta_{ij}(0) \in [0, \infty] & \text{for } i, j = 1, \dots, d. \end{cases}$$

*Suppose that there exist positive integers  $l_n, r_n$  satisfying*

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{l_n}{r_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{r_n}{n} = 0,$$

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{n^2}{r_n} \sum_{i,j=1}^d \sum_{s=l_n}^n |\rho_{ij}(s, n)| \exp\left(-\frac{2 \ln n - \ln \ln n}{1 + |\rho_{ij}(s, n)|}\right) = 0$$

*and*

$$(2.4) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i,j=1}^d \sum_{s=m}^{r_n} n^{-\frac{1-\rho_{ij}(s,n)}{1+\rho_{ij}(s,n)}} \frac{(\ln n)^{-\rho_{ij}(s,n)/(1+\rho_{ij}(s,n))}}{\sqrt{1-\rho_{ij}^2(s,n)}} = 0.$$

*Then*

$$(2.5) \quad \lim_{n \rightarrow \infty} \mathbb{P}\left(\max_{1 \leq k \leq n} X_{n,k}^{(1)} \leq u_n(x_1), \dots, \max_{1 \leq k \leq n} X_{n,k}^{(d)} \leq u_n(x_d)\right) = \exp\left(-\sum_{i=1}^d \vartheta_i(\mathbf{x}) e^{-x_i}\right), \quad \forall \mathbf{x} \in \mathbb{R}^d,$$

*where*

$$(2.6) \quad \begin{aligned} \vartheta_1(\mathbf{x}) = & \mathbb{P}\left(\frac{A}{2} + \sqrt{\delta_{t1}(k-1)} W_{k,1}^{(t)} \leq \delta_{t1}(k-1) + \frac{x_t - x_1}{2}, 1 \leq t \leq d, \right. \\ & \left. \text{for all } k \geq 2 \text{ such that } \delta_{t1}(k-1) < \infty\right) \end{aligned}$$

*and for  $i = 2, \dots, d$*

$$(2.7) \quad \begin{aligned} \vartheta_i(\mathbf{x}) = & \mathbb{P}\left(\frac{A}{2} + \sqrt{\delta_{si}(0)} W_{1,i}^{(s)} \leq \delta_{si}(0) + \frac{x_s - x_i}{2}, 1 \leq s < i, \delta_{si}(0) < \infty, \right. \\ & \frac{A}{2} + \sqrt{\delta_{ti}(k-1)} W_{k,i}^{(t)} \leq \delta_{ti}(k-1) + \frac{x_t - x_i}{2}, 1 \leq t \leq d, \\ & \left. \text{for all } k \geq 2 \text{ such that } \delta_{ti}(k-1) < \infty\right), \end{aligned}$$

*where  $\{W_{k,i}^{(t)}, 1 \leq t \leq d, \delta_{ti}(k-1) < \infty, k \geq 1\}$  are jointly normal with zero means and for each  $i = 1, \dots, d$*

$$(2.8) \quad \text{Cov}(W_{k,i}^{(j)}, W_{l,i}^{(t)}) = \frac{\delta_{ji}(k-1) + \delta_{ti}(l-1) - \delta_{jt}(|k-l|)}{2\sqrt{\delta_{ji}(k-1)\delta_{ti}(l-1)}}, \quad j, t = 1, \dots, d, \quad k, l \geq 1.$$

**Remark 2.1.** i) The  $\vartheta$ 's above should be set to 1 if all  $\delta$ 's involved are equal to infinity. Clearly, if only a finite number of  $\delta$ 's is not equal to infinity, then  $\vartheta$ 's are all positive and thus the limit in (2.5) is a max-stable distribution function. As mentioned in Remark 2 of French and Davis (2013), for some tractable correlation functions it is possible to show that  $\vartheta$ 's are positive.

ii) Note that  $\vartheta_i(\mathbf{x})$  does not depend on  $x_i$  for each  $i \leq d$ . In the particular case that  $\vartheta_i(\mathbf{x})$  is a non-degenerate distribution function, then clearly  $G(\mathbf{x}) = e^{-\sum_{i=1}^d \vartheta_i(\mathbf{x})e^{-x_i}}$  is a max-stable  $d$ -dimensional distribution function.

If condition (2.1) holds with  $\delta_{ij}(k) = \infty$  for any index  $i, j \leq d$  and  $k \geq 1$ , then clearly

$$\vartheta_i(\mathbf{x}) = \vartheta_i(x_1, \dots, x_i), \quad \mathbf{x} \in \mathbb{R}^d, i \leq d.$$

Moreover for this case the limiting distribution  $G(\mathbf{x}) = e^{-\sum_{i=1}^d \vartheta_i(x_1, \dots, x_i)e^{-x_i}}$  coincides with the  $d$ -dimensional max-stable Hüsler-Reiss distribution.

iii) As in Theorem 2.2 of Hsing, Hüsler and Reiss (1996), conditions (2.2), (2.3) and (2.4) can be replaced by

$$\lim_{n \rightarrow \infty} \sum_{1 \leq i, j \leq d} \max_{l_n \leq k \leq n} |\rho_{ij}(k, n)| \ln n = 0 \quad \text{for some } l_n = o(n)$$

and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i, j=1}^d \sum_{s=m}^{l_n} n^{-\frac{1-\rho_{ij}(s, n)}{1+\rho_{ij}(s, n)}} \frac{(\ln n)^{-\rho_{ij}(s, n)/(1+\rho_{ij}(s, n))}}{\sqrt{1-\rho_{ij}^2(s, n)}} = 0.$$

These last two conditions are easier to check than those in Theorem 2.1.

### 3. PROOFS

For notational simplicity we shall define

$$M_{k,l}^{(i)} = \max_{k < s \leq l} X_{n,s}^{(i)}, \quad M_l^{(i)} = M_{0,l}^{(i)} = \max_{1 \leq s \leq l} X_{n,s}^{(i)}, \quad M_{l,l}^{(i)} = -\infty$$

for  $i = 1, 2, \dots, d$ ,  $k = 1, \dots, l$  and  $l = 1, \dots, n$ . Before proving the theorem, we need some lemmas.

**Lemma 3.1.** For any  $n \times d$  random matrix  $\{X_{n,k}^{(i)}, 1 \leq k \leq n, 1 \leq i \leq d\}$  and any vector of constants  $(u^{(1)}, \dots, u^{(d)})$  we have

$$\begin{aligned} (3.1) \quad \mathbb{P} \left( \bigcup_{i=1}^d \{M_n^{(i)} > u^{(i)}\} \right) &= \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(1)} > u^{(1)}, \bigcap_{t=1}^d \{M_{k,n}^{(t)} \leq u^{(t)}\} \right) \\ &\quad + \sum_{i=2}^d \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(i)} > u^{(i)}, \bigcap_{s=1}^{i-1} \{M_{k-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=i}^d \{M_{k,n}^{(t)} \leq u^{(t)}\} \right). \end{aligned}$$

*Proof.* The case of  $d = 1$  directly follows from O'Brien (1987). We shall prove the case of  $d = 2$  and then use the induction method to conclude that the lemma holds for any  $d \geq 2$ .

It is straightforward to check that for any  $s \geq 0$  and  $i = 1, \dots, d$ ,

$$\begin{aligned} \mathbb{P} \left( M_{s,n}^{(i)} > u^{(i)} \right) &= \mathbb{P} \left( X_{n,n}^{(i)} > u^{(i)} \right) + \mathbb{P} \left( M_{s,n-1}^{(i)} > u^{(i)}, X_{n,n}^{(i)} \leq u^{(i)} \right) \\ &= \mathbb{P} \left( X_{n,n}^{(i)} > u^{(i)}, M_{n,n}^{(i)} \leq u^{(i)} \right) + \mathbb{P} \left( X_{n,n-1}^{(i)} > u^{(i)}, X_{n,n}^{(i)} \leq u^{(i)} \right) \\ &\quad + \mathbb{P} \left( M_{s,n-2}^{(i)} > u^{(i)}, X_{n,n-1}^{(i)} \leq u^{(i)}, X_{n,n}^{(i)} \leq u^{(i)} \right) \\ &= \mathbb{P} \left( X_{n,n}^{(i)} > u^{(i)}, M_{n,n}^{(i)} \leq u^{(i)} \right) + \mathbb{P} \left( X_{n,n-1}^{(i)} > u^{(i)}, M_{n-1,n}^{(i)} \leq u^{(i)} \right) \end{aligned}$$

$$+\mathbb{P}\left(M_{s,n-2}^{(i)} > u^{(i)}, X_{n,n-1}^{(i)} \leq u^{(i)}, X_{n,n}^{(i)} \leq u^{(i)}\right).$$

Continuing the above decomposition, we have

$$(3.2) \quad \mathbb{P}\left(M_{s,n}^{(i)} > u^{(i)}\right) = \sum_{k=s+1}^n \mathbb{P}\left(X_{n,k}^{(i)} > u^{(i)}, M_{k,n}^{(i)} \leq u^{(i)}\right)$$

for any  $s \geq 0$ . For proving that (3.1) holds for the case of  $d = 2$ , we first note that

$$\begin{aligned} & \mathbb{P}\left(M_n^{(1)} \leq u^{(1)}, M_n^{(2)} > u^{(2)}\right) \\ &= \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_n^{(1)} \leq u^{(1)}\right) \\ &= \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) \\ &\quad - \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}, M_{k-1}^{(1)} > u^{(1)}\right) \\ &= \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) \\ &\quad - \sum_{k=1}^n \sum_{l=1}^{k-1} \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right) \\ &= \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) \\ &\quad - \sum_{l=1}^{n-1} \sum_{k=l+1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right) \\ (3.3) \quad &= \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) - \sum_{l=1}^{n-1} \mathbb{P}\left(M_{l,n}^{(2)} > u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right), \end{aligned}$$

which can be used to show that

$$\begin{aligned} & \mathbb{P}\left(\{M_n^{(1)} > u^{(1)}\} \cup \{M_n^{(2)} > u^{(2)}\}\right) \\ &= \mathbb{P}\left(M_n^{(1)} > u^{(1)}\right) + \mathbb{P}\left(M_n^{(1)} \leq u^{(1)}, M_n^{(2)} > u^{(2)}\right) \\ &= \sum_{l=1}^n \mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right) + \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) \\ &\quad - \sum_{l=1}^{n-1} \mathbb{P}\left(M_{l,n}^{(2)} > u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right) \\ &= \mathbb{P}\left(X_{n,n}^{(1)} > u^{(1)}\right) + \left(\sum_{l=1}^{n-1} \mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right) - \sum_{l=1}^{n-1} \mathbb{P}\left(M_{l,n}^{(2)} > u^{(2)}, X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}\right)\right) \\ &\quad + \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) \\ &= \mathbb{P}\left(X_{n,n}^{(1)} > u^{(1)}\right) + \sum_{l=1}^{n-1} \mathbb{P}\left(X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}, M_{l,n}^{(2)} \leq u^{(2)}\right) \\ &\quad + \sum_{k=1}^n \mathbb{P}\left(X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)}\right) \end{aligned}$$

$$= \sum_{l=1}^n \mathbb{P} \left( X_{n,l}^{(1)} > u^{(1)}, M_{l,n}^{(1)} \leq u^{(1)}, M_{l,n}^{(2)} \leq u^{(2)} \right) + \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(2)} > u^{(2)}, M_{k,n}^{(2)} \leq u^{(2)}, M_{k-1,n}^{(1)} \leq u^{(1)} \right),$$

i.e., (3.1) holds for  $d = 2$ .

Next, suppose that (3.1) holds for  $d = m - 1 > 2$ , i.e.,

$$(3.4) \quad \mathbb{P} \left( \bigcup_{i=1}^{m-1} \{M_n^{(i)} > u^{(i)}\} \right) = \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(1)} > u^{(1)}, \bigcap_{t=1}^{m-1} \{M_{k,n}^{(t)} \leq u^{(t)}\} \right) \\ + \sum_{i=2}^{m-1} \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(i)} > u^{(i)}, \bigcap_{s=1}^{i-1} \{M_{k-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=i}^{m-1} \{M_{k,n}^{(t)} \leq u^{(t)}\} \right).$$

In view of (3.2) and (3.4), we have

$$(3.5) \quad \mathbb{P} \left( \bigcap_{i=1}^{m-1} \{M_n^{(i)} \leq u^{(i)}\}, M_n^{(m)} > u^{(m)} \right) \\ = \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_n^{(i)} \leq u^{(i)}\} \right) \\ = \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\ - \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\}, \bigcup_{j=1}^{m-1} \{M_{k-1}^{(j)} > u^{(j)}\} \right) \\ = \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\ - \sum_{k=1}^n \sum_{l=1}^{k-1} \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\}, X_{n,l}^{(1)} > u^{(1)}, \bigcap_{t=1}^{m-1} \{M_{l,k-1}^{(t)} \leq u^{(t)}\} \right) \\ - \sum_{k=1}^n \sum_{j=2}^{m-1} \sum_{l=1}^{k-1} \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\}, \right. \\ \left. X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,k-1}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^{m-1} \{M_{l,k-1}^{(t)} \leq u^{(t)}\} \right) \\ = \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\ - \sum_{l=1}^{n-1} \sum_{k=l+1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{l,n}^{(i)} \leq u^{(i)}\}, X_{n,l}^{(1)} > u^{(1)} \right) \\ - \sum_{j=2}^{m-1} \sum_{l=1}^{n-1} \sum_{k=l+1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^{m-1} \{M_{l,n}^{(t)} \leq u^{(t)}\} \right) \\ = \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\ - \sum_{l=1}^{n-1} \mathbb{P} \left( M_{l,n}^{(m)} > u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{l,n}^{(i)} \leq u^{(i)}\}, X_{n,l}^{(1)} > u^{(1)} \right) \\ - \sum_{j=2}^{m-1} \sum_{l=1}^{n-1} \mathbb{P} \left( X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^{m-1} \{M_{l,n}^{(t)} \leq u^{(t)}\}, M_{l,n}^{(m)} > u^{(m)} \right).$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
& \mathbb{P} \left( \bigcup_{i=1}^m \{M_n^{(i)} > u^{(i)}\} \right) \\
&= \mathbb{P} \left( \bigcup_{i=1}^{m-1} \{M_n^{(i)} > u^{(i)}\} \right) + \mathbb{P} \left( \bigcap_{i=1}^{m-1} \{M_n^{(i)} \leq u^{(i)}\}, M_n^{(m)} > u^{(m)} \right) \\
&= \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(1)} > u^{(1)}, \bigcap_{t=1}^{m-1} \{M_{k,n}^{(t)} \leq u^{(t)}\} \right) \\
&\quad + \sum_{i=2}^{m-1} \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(i)} > u^{(i)}, \bigcap_{s=1}^{i-1} \{M_{k-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=i}^{m-1} \{M_{k,n}^{(t)} \leq u^{(t)}\} \right) \\
&\quad + \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\
&\quad - \sum_{l=1}^{n-1} \mathbb{P} \left( M_{l,n}^{(m)} > u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{l,n}^{(i)} \leq u^{(i)}\}, X_{n,l}^{(1)} > u^{(1)} \right) \\
&\quad - \sum_{j=2}^{m-1} \sum_{l=1}^{n-1} \mathbb{P} \left( X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^{m-1} \{M_{l,n}^{(t)} \leq u^{(t)}\}, M_{l,n}^{(m)} > u^{(m)} \right) \\
&= \left( \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(1)} > u^{(1)}, \bigcap_{t=1}^{m-1} \{M_{k,n}^{(t)} \leq u^{(t)}\} \right) - \sum_{l=1}^{n-1} \mathbb{P} \left( M_{l,n}^{(m)} > u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{l,n}^{(i)} \leq u^{(i)}\}, X_{n,l}^{(1)} > u^{(1)} \right) \right) \\
&\quad + \left( \sum_{i=2}^{m-1} \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(i)} > u^{(i)}, \bigcap_{s=1}^{i-1} \{M_{k-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=i}^{m-1} \{M_{k,n}^{(t)} \leq u^{(t)}\} \right) \right. \\
&\quad \left. - \sum_{j=2}^{m-1} \sum_{l=1}^{n-1} \mathbb{P} \left( X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^{m-1} \{M_{l,n}^{(t)} \leq u^{(t)}\}, M_{l,n}^{(m)} > u^{(m)} \right) \right) \\
&\quad + \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\
&= \left( \mathbb{P} \left( X_{n,n}^{(1)} > u^{(1)} \right) + \sum_{l=1}^{n-1} \mathbb{P} \left( X_{n,l}^{(1)} > u^{(1)}, \bigcap_{i=1}^{m-1} \{M_{l,n}^{(i)} \leq u^{(i)}\}, M_{l,n}^{(m)} \leq u^{(m)} \right) \right) \\
&\quad + \left( \sum_{i=2}^{m-1} \mathbb{P} \left( X_{n,n}^{(i)} > u^{(i)}, \bigcap_{s=1}^{i-1} \{X_{n,n}^{(s)} \leq u^{(s)}\} \right) \right. \\
&\quad \left. + \sum_{j=2}^{m-1} \sum_{l=1}^{n-1} \mathbb{P} \left( X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^{m-1} \{M_{l,n}^{(t)} \leq u^{(t)}\}, M_{l,n}^{(m)} \leq u^{(m)} \right) \right) \\
&\quad + \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\
&= \sum_{l=1}^n \mathbb{P} \left( X_{n,l}^{(1)} > u^{(1)}, \bigcap_{i=1}^m \{M_{l,n}^{(i)} \leq u^{(i)}\} \right) + \sum_{j=2}^{m-1} \sum_{l=1}^n \mathbb{P} \left( X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^m \{M_{l,n}^{(t)} \leq u^{(t)}\} \right) \\
&\quad + \sum_{k=1}^n \mathbb{P} \left( X_{n,k}^{(m)} > u^{(m)}, M_{k,n}^{(m)} \leq u^{(m)}, \bigcap_{i=1}^{m-1} \{M_{k-1,n}^{(i)} \leq u^{(i)}\} \right) \\
&= \sum_{l=1}^n \mathbb{P} \left( X_{n,l}^{(1)} > u^{(1)}, \bigcap_{i=1}^m \{M_{l,n}^{(i)} \leq u^{(i)}\} \right)
\end{aligned}$$

$$+ \sum_{j=2}^m \sum_{l=1}^n \mathbb{P} \left( X_{n,l}^{(j)} > u^{(j)}, \bigcap_{s=1}^{j-1} \{M_{l-1,n}^{(s)} \leq u^{(s)}\}, \bigcap_{t=j}^m \{M_{l,n}^{(t)} \leq u^{(t)}\} \right),$$

i.e., (3.1) holds for  $d = m$ . Hence the lemma follows from the induction method.  $\square$

**Lemma 3.2.** *Let  $\{X_{n,k}, k, n \geq 1\}$  be a  $d$ -dimensional stationary Gaussian triangular array. If there exist positive integers  $l_n$  and  $r_n$  such that (2.2) and (2.3) hold, then we have for any  $x_i \in \mathbb{R}, i \leq d$*

$$(3.6) \quad \lim_{n \rightarrow \infty} \left( \mathbb{P} \left( \bigcap_{i=1}^d \{M_n^{(i)} \leq u_n(x_i)\} \right) - \left( \mathbb{P} \left( \bigcap_{i=1}^d \{M_{r_n}^{(i)} \leq u_n(x_i)\} \right) \right)^{q_n} \right) = 0,$$

where  $q_n = \lfloor n/r_n \rfloor$ .

*Proof.* Define  $N_n = \{1, 2, \dots, n\}$  for any positive integer  $n$  and set

$$N_{r_n q_n} = (I_1 \cup J_1) \cup (I_2 \cup J_2) \cup \dots \cup (I_{q_n} \cup J_{q_n}),$$

with  $I_s = \{(s-1)r_n + 1, \dots, sr_n - l_n\}$  and  $J_s = \{sr_n - l_n + 1, \dots, sr_n\}$  for  $s = 1, 2, \dots, q_n$ . Since  $r_n q_n \leq n < (r_n + 1)q_n < r_n q_n + l_n$ , we get  $|N_n \setminus N_{r_n q_n}| < q_n < l_n$ , where  $|K|$  means the length of the interval  $K \subset \mathbb{R}$ . Further, define sets  $I_{q_n+1}$  and  $J_{q_n+1}$  by

$$I_{q_n+1} = \{r_n q_n - r_n + l_n + 1, \dots, r_n q_n - 1, r_n q_n\},$$

$$J_{q_n+1} = \{r_n q_n + 1, \dots, r_n q_n + l_n - 1, r_n q_n + l_n\}.$$

Clearly,  $|I_{q_n+1}| = r_n - l_n$ ,  $|J_{q_n+1}| = l_n$  and  $I_{q_n+1} \subset N_{r_n q_n}$  and  $N_n \setminus N_{r_n q_n} \subset J_{q_n+1}$ . Using the fact that

$$l_n = o(r_n), \quad l_n = o(n), \quad \lim_{n \rightarrow \infty} n(1 - \Phi(u_n(x_i))) = e^{-x_i}$$

we obtain

$$\begin{aligned} 0 &\leq \mathbb{P} \left( \bigcap_{s=1}^{q_n} \bigcap_{i=1}^d \{M^{(i)}(I_s) \leq u_n(x_i)\} \right) - \mathbb{P} \left( \bigcap_{i=1}^d \{M_n^{(i)} \leq u_n(x_i)\} \right) \\ &\leq \sum_{s=1}^{q_n+1} \sum_{i=1}^d \mathbb{P} \left( M^{(i)}(I_s) \leq u_n(x_i) < M^{(i)}(J_s) \right) \\ &\leq \sum_{s=1}^{q_n+1} \sum_{i=1}^d \mathbb{P} \left( u_n(x_i) < M^{(i)}(J_s) \right) \\ &\leq (q_n + 1) l_n \sum_{i=1}^d (1 - \Phi(u_n(x_i))) \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , where  $M^{(i)}(I_s) = \max_{j \in I_s} X_{n,j}^{(i)}$ . Using Berman's inequality given in Li and Shao (2001) (see also Piterbarg (1996)) and (2.3)

$$\begin{aligned} &\left| \mathbb{P} \left( \bigcap_{s=1}^{q_n} \bigcap_{i=1}^d \{M^{(i)}(I_s) \leq u_n(x_i)\} \right) - \prod_{s=1}^{q_n} \mathbb{P} \left( \bigcap_{i=1}^d \{M^{(i)}(I_s) \leq u_n(x_i)\} \right) \right| \\ &\leq (q_n - 1) \frac{1}{2\pi} \sum_{i,j=1}^d \sum_{1 \leq s < t \leq n, t-s > l_n} \left| \arcsin(\rho_{ij}(t-s, n)) \right| \exp \left( -\frac{u_n^2(x_i) + u_n^2(x_j)}{2(1 + |\rho_{ij}(t-s, n)|)} \right) \\ &\leq C \frac{n^2}{r_n} \sum_{i,j=1}^d \sum_{s=l_n}^n |\rho_{ij}(s, n)| \exp \left( -\frac{2 \ln n - \ln \ln n}{1 + |\rho_{ij}(s, n)|} \right) \end{aligned}$$



$$\rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where  $C$  is some positive constant. Since further

$$\begin{aligned} 0 &\leq \prod_{s=1}^{q_n} \mathbb{P} \left( \bigcap_{i=1}^d \{M^{(i)}(I_s) \leq u_n(x_i)\} \right) - \prod_{s=1}^{q_n} \mathbb{P} \left( \bigcap_{i=1}^d \{M^{(i)}(I_s \cup J_s) \leq u_n(x_i)\} \right) \\ &\leq \sum_{s=1}^{q_n} \sum_{i=1}^d \mathbb{P} \left( u_n(x_i) < M^{(i)}(J_s) \right) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , the lemma follows.  $\square$

**Remark 3.1.** If  $\{s_n, n \geq 1\}$  is a sequence of positive integers such that  $s_n = o(n)$  and  $r_n = o(s_n)$ , then clearly both (2.2) and (2.3) hold with  $r_n$  replaced by  $s_n$ . From the proof above we see that these two conditions are the only assumptions of Lemma 3.2. Hence (3.6) still holds if we substitute  $q_n$  by  $t_n = [n/s_n]$ .

**Lemma 3.3.** Under the conditions of Theorem 2.1, for any bounded index set  $K \subset \{2, 3, \dots\}$  and each  $c \in \{2, \dots, d\}$  we have

$$\begin{aligned} (3.7) \quad &\lim_{n \rightarrow \infty} \mathbb{P} \left( X_{n,1}^{(s)} \leq u_n(x_s), 1 \leq s < c, X_{n,k}^{(t)} \leq u_n(x_t), 1 \leq t \leq d, k \in K | X_{n,1}^{(c)} > u_n(x_c) \right) \\ &= \mathbb{P} \left( \frac{A}{2} + \sqrt{\delta_{sc}(0)} W_{1,c}^{(s)} \leq \delta_{sc}(0) + \frac{x_s - x_c}{2}, 1 \leq s < c, \delta_{sc}(0) < \infty, \right. \\ &\quad \left. \frac{A}{2} + \sqrt{\delta_{tc}(k-1)} W_{k,c}^{(t)} \leq \delta_{tc}(k-1) + \frac{x_t - x_c}{2}, 1 \leq t \leq d, \text{ for all } k \in K \text{ such that } \delta_{tc}(k-1) < \infty \right). \end{aligned}$$

Further

$$\begin{aligned} (3.8) \quad &\lim_{n \rightarrow \infty} \mathbb{P} \left( X_{n,k}^{(t)} \leq u_n(x_t), 1 \leq t \leq d, k \in K | X_{n,1}^{(1)} > u_n(x_1) \right) \\ &= \mathbb{P} \left( \frac{A}{2} + \sqrt{\delta_{t1}(k-1)} W_{k,1}^{(t)} \leq \delta_{t1}(k-1) + \frac{x_t - x_1}{2}, 1 \leq t \leq d, \text{ for all } k \in K \text{ such that } \delta_{t1}(k-1) < \infty \right), \end{aligned}$$

and  $\{W_{k,i}^{(t)}, 1 \leq t \leq d, \delta_{ti}(k-1) < \infty, k \in \{1\} \cup K\}$  are jointly normal with zero means and

$$\text{Cov}(W_{k,i}^{(j)}, W_{l,i}^{(t)}) = \frac{\delta_{ji}(k-1) + \delta_{ti}(l-1) - \delta_{jt}(|k-l|)}{2\sqrt{\delta_{ji}(k-1)\delta_{ti}(l-1)}}, \quad i, j, t = 1, \dots, d, \quad k, l \in \{1\} \cup K.$$

*Proof.* We follow the arguments in the proof of Lemma 4.1 in Hsing, Hüsler and Reiss (1996). First like (4.1) therein we have for each  $c \in \{2, \dots, d\}$ ,

$$\begin{aligned} (3.9) \quad &\mathbb{P} \left( X_{n,1}^{(s)} \leq u_n(x_s), 1 \leq s < c, X_{n,k}^{(t)} \leq u_n(x_t), 1 \leq t \leq d, k \in K | X_{n,1}^{(c)} > u_n(x_c) \right) \\ &\sim \int_0^\infty \mathbb{P} \left( X_{n,1}^{(s)} \leq u_n(x_s), 1 \leq s < c, X_{n,k}^{(t)} \leq u_n(x_t), 1 \leq t \leq d, k \in K | X_{n,1}^{(c)} = T_n(x_c, z) \right) \\ &\quad \times \exp \left( -z - \frac{z^2}{2u_n^2(x_c)} \right) dz, \end{aligned}$$

where  $T_n(x_c, z) = u_n(x_c) + z/u_n(x_c)$ . Let  $\{Y_{n,k,c}^{(i)}, 1 \leq i \leq d, k \in \{1\} \cup K\}$  have the same distribution as the conditional distribution of  $\{X_{n,k}^{(i)}, 1 \leq i \leq d, k \in \{1\} \cup K\}$  given  $X_{n,1}^{(c)} = T_n(x_c, z)$ . Then

$$\mathbb{E} \left\{ Y_{n,k,c}^{(i)} \right\} = \rho_{ic}(k-1, n) T_n(x_c, z)$$

and

$$\text{Cov}(Y_{n,k,c}^{(i)}, Y_{n,l,c}^{(j)}) = \rho_{ij}(|k-l|, n) - \rho_{ic}(k-1, n) \rho_{jc}(l-1, n)$$

for  $i, j \in \{1, \dots, d\}$  and  $k, l \in \{1\} \cup K$ . Further define

$$Z_{n,k,c}^{(i)} = \frac{Y_{n,k,c}^{(i)} - \rho_{ic}(k-1, n) T_n(x_c, z)}{\sqrt{1 - \rho_{ic}^2(k-1, n)}}, \quad 1 \leq i \leq d, k \in \{1\} \cup K.$$

Then we have

$$\text{Cov}(Z_{n,k,c}^{(i)}, Z_{n,l,c}^{(j)}) = \frac{\rho_{ij}(|k-l|, n) - \rho_{ic}(k-1, n)\rho_{jc}(l-1, n)}{\sqrt{(1-\rho_{ic}^2(k-1, n))(1-\rho_{jc}^2(l-1, n))}} \rightarrow \frac{\delta_{ic}(k-1) + \delta_{jc}(l-1) - \delta_{ij}(|k-l|)}{2\sqrt{\delta_{ic}(k-1)\delta_{jc}(l-1)}},$$

for  $i, j \in \{1, \dots, d\}$ ,  $k, l \in \{1\} \cup K$ . Thus, using  $u_n^2(x) \sim 2 \ln n$  for  $x \in \mathbb{R}$  we have

$$\begin{aligned} & \mathbb{P}\left(Y_{n,1,c}^{(s)} \leq u_n(x_s), 1 \leq s < c, Y_{n,k,c}^{(t)} \leq u_n(x_t), 1 \leq t \leq d, k \in K\right) \\ = & \mathbb{P}\left(\frac{1}{2}\rho_{sc}(0, n)z + \sqrt{\frac{1+\rho_{sc}(0, n)}{2}}\sqrt{\frac{u_n^2(x_c)(1-\rho_{sc}(0, n))}{2}}Z_{n,1,c}^{(s)} \leq \frac{1}{2}(u_n(x_s)u_n(x_c) - \rho_{sc}(0, n)u_n^2(x_c)), \right. \\ & \frac{1}{2}\rho_{tc}(k-1, n)z + \sqrt{\frac{1+\rho_{tc}(k-1, n)}{2}}\sqrt{\frac{u_n^2(x_c)(1-\rho_{tc}(k-1, n))}{2}}Z_{n,k,c}^{(t)} \\ (3.10) \quad & \leq \frac{1}{2}(u_n(x_t)u_n(x_c) - \rho_{tc}(k-1, n)u_n^2(x_c)), \quad \text{for } 1 \leq s < c, 1 \leq t \leq d, k \in K) \\ \rightarrow & \mathbb{P}\left(\frac{z}{2} + \sqrt{\delta_{sc}(0)}W_{1,c}^{(s)} \leq \delta_{sc}(0) + \frac{x_s - x_c}{2}, 1 \leq s < c, \delta_{sc}(0) < \infty, \right. \\ & \frac{z}{2} + \sqrt{\delta_{tc}(k-1)}W_{k,c}^{(t)} \leq \delta_{tc}(k-1) + \frac{x_t - x_c}{2}, 1 \leq t \leq d, \\ & \left. \text{for all } k \in K \text{ such that } \delta_{tc}(k-1) < \infty\right). \end{aligned}$$

Therefore, (3.7) follows by (3.10) and (3.9). The proof of (3.8) can be established with similar arguments. Hence the claim follows.  $\square$

**Lemma 3.4.** *Under the conditions of Theorem 2.1, for  $c \in \{1, \dots, d\}$  we have*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{i=1}^d \bigcup_{j=m}^{r_n} \{X_{n,j}^{(i)} > u_n(x_i)\} \mid X_{n,1}^{(c)} > u_n(x_c)\right) = 0.$$

*Proof.* It suffices to show that for each fixed  $i \in \{1, \dots, d\}$

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{j=m}^{r_n} \{X_{n,j}^{(i)} > u_n(x_i)\} \mid X_{n,1}^{(c)} > u_n(x_c)\right) = 0.$$

As in the proof of Lemma 3.3, write with  $a_{nj}(z) = \rho_{ic}(j-1, n)(u_n(x_c) + z/u_n(x_c))$  and  $b_{nj} := \sqrt{1 - \rho_{ic}^2(j-1, n)}$

$$\mathbb{P}\left(\bigcup_{j=m}^{r_n} \{X_{n,j}^{(i)} > u_n(x_i)\} \mid X_{n,1}^{(c)} > u_n(x_c)\right) \sim \int_0^\infty \mathbb{P}\left(\bigcup_{j=m}^{r_n} \{a_{nj}(z) + Z_{n,j,c}^{(i)} b_{nj} > u_n(x_i)\}\right) \exp\left(-z - \frac{z^2}{2u_n^2(x_c)}\right) dz.$$

Hence, we only need to show that for each fixed  $z_0 > 0$

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_0^{z_0} \mathbb{P}\left(\bigcup_{j=m}^{r_n} \{a_{nj}(z) + Z_{n,j,c}^{(i)} b_{nj} > u_n(x_i)\}\right) \exp\left(-z - \frac{z^2}{2u_n^2(x_c)}\right) dz = 0,$$

which follows if we show

$$(3.11) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{0 \leq z \leq z_0} \sum_{j=m}^{r_n} \mathbb{P}\left(a_{nj}(z) + Z_{n,j,c}^{(i)} b_{nj} > u_n(x_i)\right) = 0.$$

In view of the derivation of (4.4) in Hsing, Hüsler and Reiss (1996), condition (2.4) implies

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{m \leq j \leq r_n} ((1 - \rho_{ic}(j-1, n)) \ln n)^{-1} = 0.$$

Thus, for large  $n$  and  $j \in [m, r_n]$  we have

$$\theta_{nj} := \frac{u_n(x_i) - u_n(x_c)\rho_{ic}(j-1, n)}{\sqrt{1 - \rho_{ic}^2(j-1, n)}} - \frac{z\rho_{ic}(j-1, n)}{u_n(x_c)\sqrt{1 - \rho_{ic}^2(j-1, n)}} > 0.$$

By the fact that  $1 - \Phi(x) \leq x^{-1}\varphi(x)$  for  $x > 0$ , we obtain

$$\mathbb{P}\left(Z_{n,j,c}^{(i)} > \theta_{nj}\right) \leq \frac{1}{\theta_{nj}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\theta_{nj}^2\right).$$

Next, for some positive constant  $C$  depending only on  $x_i, x_c$  and  $z_0$  we have

$$\begin{aligned}\theta_{nj}^2 &\leq C + \frac{1 - \rho_{ic}(j-1, n)}{1 + \rho_{ic}(j-1, n)} b_n^2 \\ &\leq C + \frac{1 - \rho_{ic}(j-1, n)}{1 + \rho_{ic}(j-1, n)} (2 \ln n - \ln \ln n),\end{aligned}$$

which implies that

$$(3.12) \quad \mathbb{P}\left(Z_{n,j,c}^{(i)} > \theta_{nj}\right) \leq C^* b_{nj}^{-1} n^{-\frac{1-\rho_{ic}(j-1,n)}{1+\rho_{ic}(j-1,n)}} (\ln n)^{-\frac{\rho_{ic}(j-1,n)}{1+\rho_{ic}(j-1,n)}}$$

for some  $C^*$  depending on  $x_i, x_c$  and  $z_0$ . Hence (3.11) follows from (3.12), i.e., the lemma holds.  $\square$

*Proof of Theorem 2.1.* In view of Lemma 3.3

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{t=1}^d \{M_{1,m}^{(t)} \leq u_n(x_t)\} \mid X_{n,1}^{(1)} > u_n(x_1)\right) = \vartheta_1(\mathbf{x})$$

and for  $i = 2, \dots, d$

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{s=1}^{i-1} \{M_m^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,m}^{(t)} \leq u_n(x_t)\} \mid X_{n,1}^{(i)} > u_n(x_i)\right) = \vartheta_i(\mathbf{x}),$$

with  $\vartheta_1(\mathbf{x})$  and  $\vartheta_i(\mathbf{x})$  defined in (2.6) and (2.7) respectively, and by making use of Lemma 3.4

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{t=1}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \mid X_{n,1}^{(1)} > u_n(x_1)\right) = \vartheta_1(\mathbf{x})$$

and for  $i = 2, \dots, d$

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{s=1}^{i-1} \{M_{r_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \mid X_{n,1}^{(i)} > u_n(x_i)\right) = \vartheta_i(\mathbf{x}).$$

Hence, by  $n(1 - \Phi(u_n(x))) \rightarrow e^{-x}$  as  $n \rightarrow \infty$ , the theorem follows if further

$$\begin{aligned}&\mathbb{P}\left(\bigcap_{i=1}^d \{M_n^{(i)} \leq u_n(x_i)\}\right) \\ &- \exp\left(-n \mathbb{P}\left(X_{n,1}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\}\right)\right. \\ &\quad \left.- n \sum_{i=2}^d \mathbb{P}\left(X_{n,1}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{r_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\}\right)\right) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Following the arguments in the proof of Theorem 2.1 in O'Brien (1987), we first derive an asymptotic upper bound for  $p_{n,d} := \mathbb{P}\left(\bigcap_{i=1}^d \{M_n^{(i)} \leq u_n(x_i)\}\right)$ . Utilising (3.6) and Lemma 3.1 for all large  $n$  we obtain

$$\begin{aligned}p_{n,d} &= \left(\mathbb{P}\left(\bigcap_{i=1}^d \{M_{r_n}^{(i)} \leq u_n(x_i)\}\right)\right)^{q_n} + o(1) \\ &= \left(1 - \mathbb{P}\left(\bigcup_{i=1}^d \{M_{r_n}^{(i)} > u_n(x_i)\}\right)\right)^{q_n} + o(1) \\ &\leq \left(1 - r_n \mathbb{P}\left(X_{n,1}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\}\right)\right)\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=2}^d r_n \mathbb{P} \left( X_{n,1}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{r_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \Big)^{q_n} + o(1) \\
& \leq \exp \left( -n \mathbb{P} \left( X_{n,1}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \right. \\
& \quad \left. -n \sum_{i=2}^d \mathbb{P} \left( X_{n,1}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{r_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \right) + o(1).
\end{aligned}$$

The rest of the proof is dedicated to the derivation of an asymptotic lower bound for  $p_{n,d}$ . Choose a sequence of positive integers  $\{s_n, n \geq 1\}$  such that  $r_n = o(s_n)$ ,  $s_n = o(n)$ , and (3.6) holds with  $r_n$  replaced by  $s_n$  and  $q_n$  replaced by  $t_n = \lfloor n/s_n \rfloor$ . In view of the assumptions (see Remark 3.1) this is possible. Since  $r_n = o(s_n)$ , we have

$$(3.13) \quad \mathbb{P} \left( M_{r_n}^{(i)} > u_n(x_i) \right) = o \left( \mathbb{P} \left( M_{s_n}^{(i)} > u_n(x_i) \right) \right), \quad 1 \leq i \leq d.$$

We proceed by induction showing that as  $n \rightarrow \infty$

$$\begin{aligned}
(3.14) \quad & \mathbb{P} \left( \bigcup_{i=1}^d \{M_{s_n}^{(i)} > u_n(x_i)\} \right) \\
& = \left( \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{k,s_n}^{(t)} \leq u_n(x_t)\} \right) \right. \\
& \quad \left. + \sum_{i=2}^d \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{k-1,s_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{k,s_n}^{(t)} \leq u_n(x_t)\} \right) \right) (1 + o(1)).
\end{aligned}$$

If  $d = 1$ , as in O'Brien (1987), we have

$$\begin{aligned}
\mathbb{P} \left( M_{s_n}^{(1)} > u_n(x_1) \right) & = \mathbb{P} \left( M_{s_n-r_n}^{(1)} > u_n(x_1), M_{s_n-r_n,s_n}^{(1)} \leq u_n(x_1) \right) + \mathbb{P} \left( M_{r_n}^{(1)} > u_n(x_1) \right) \\
& = \left( \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(1)} > u_n(x_1), M_{k,s_n}^{(1)} \leq u_n(x_1) \right) \right) (1 + o(1)) \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

For  $d = 2$ , by (3.2), (3.3) and stationarity we have

$$\begin{aligned}
& \mathbb{P} \left( \{M_{s_n}^{(1)} > u_n(x_1)\} \cup \{M_{s_n}^{(2)} > u_n(x_2)\} \right) \\
& = \mathbb{P} \left( M_{s_n}^{(1)} > u_n(x_1) \right) + \mathbb{P} \left( M_{s_n}^{(2)} > u_n(x_2), M_{s_n}^{(1)} \leq u_n(x_1) \right) \\
& = \mathbb{P} \left( M_{s_n}^{(1)} > u_n(x_1) \right) + \mathbb{P} \left( M_{s_n}^{(2)} > u_n(x_2), M_{s_n}^{(1)} \leq u_n(x_1) \right) \\
& \quad - \mathbb{P} \left( M_{s_n-r_n,s_n}^{(2)} > u_n(x_2), M_{s_n-r_n,s_n}^{(1)} \leq u_n(x_1) \right) + \mathbb{P} \left( M_{r_n}^{(2)} > u_n(x_2), M_{r_n}^{(1)} \leq u_n(x_1) \right) \\
& = \left( \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(1)} > u_n(x_1), M_{k,s_n}^{(1)} \leq u_n(x_1) \right) + \sum_{k=1}^{s_n} \mathbb{P} \left( X_{n,k}^{(2)} > u_n(x_2), M_{k,s_n}^{(2)} \leq u_n(x_2), M_{k-1,s_n}^{(1)} \leq u_n(x_1) \right) \right. \\
& \quad - \sum_{k=1}^{s_n-1} \mathbb{P} \left( M_{k,s_n}^{(2)} > u_n(x_2), X_{n,k}^{(1)} > u_n(x_1), M_{k,s_n}^{(1)} \leq u_n(x_1) \right) \\
& \quad - \sum_{k=s_n-r_n+1}^{s_n} \mathbb{P} \left( X_{n,k}^{(2)} > u_n(x_2), M_{k,s_n}^{(2)} \leq u_n(x_2), M_{k-1,s_n}^{(1)} \leq u_n(x_1) \right) \\
& \quad \left. + \sum_{k=s_n-r_n+1}^{s_n-1} \mathbb{P} \left( M_{k,s_n}^{(2)} > u_n(x_2), X_{n,k}^{(1)} > u_n(x_1), M_{k,s_n}^{(1)} \leq u_n(x_1) \right) \right) (1 + o(1)) \\
& = \left( \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(1)} > u_n(x_1), M_{k,s_n}^{(1)} \leq u_n(x_1), M_{k,s_n}^{(2)} \leq u_n(x_2) \right) \right.
\end{aligned}$$

$$+ \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(2)} > u_n(x_2), M_{k,s_n}^{(2)} \leq u_n(x_2), M_{k-1,s_n}^{(1)} \leq u_n(x_1) \right) (1 + o(1)),$$

i.e., (3.14) holds for  $d = 2$ . Assume next that (3.14) holds for  $d = m - 1 > 2$ . By (3.13)

$$\begin{aligned} \mathbb{P} \left( \bigcup_{i=1}^m \{M_{s_n}^{(i)} > u_n(x_i)\} \right) &= \mathbb{P} \left( \bigcup_{i=1}^{m-1} \{M_{s_n}^{(i)} > u_n(x_i)\} \right) + \mathbb{P} \left( \bigcap_{i=1}^{m-1} \{M_{s_n}^{(i)} \leq u_n(x_i)\}, M_{s_n}^{(m)} > u_n(x_m) \right) \\ &= \left( \mathbb{P} \left( \bigcup_{i=1}^{m-1} \{M_{s_n}^{(i)} > u_n(x_i)\} \right) + \mathbb{P} \left( \bigcap_{i=1}^{m-1} \{M_{s_n}^{(i)} \leq u_n(x_i)\}, M_{s_n}^{(m)} > u_n(x_m) \right) \right. \\ &\quad \left. - \mathbb{P} \left( \bigcap_{i=1}^{m-1} \{M_{s_n-r_n}^{(i)} \leq u_n(x_i)\}, M_{s_n-r_n}^{(m)} > u_n(x_m) \right) \right) (1 + o(1)). \end{aligned}$$

Consequently (3.5) implies that (3.14) holds for  $d = m$ . According to (3.14), by stationarity we have

$$\begin{aligned} &\mathbb{P} \left( \bigcup_{i=1}^d \{M_{s_n}^{(i)} > u_n(x_i)\} \right) \\ &\leq \left( \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{k,r_n+k-1}^{(t)} \leq u_n(x_t)\} \right) \right. \\ &\quad \left. + \sum_{i=2}^d \sum_{k=1}^{s_n-r_n} \mathbb{P} \left( X_{n,k}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{k-1,r_n+k-1}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{k,r_n+k-1}^{(t)} \leq u_n(x_t)\} \right) \right) (1 + o(1)) \\ &\leq \left( s_n \mathbb{P} \left( X_{n,1}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \right. \\ &\quad \left. + \sum_{i=2}^d s_n \mathbb{P} \left( X_{n,1}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{r_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \right) (1 + o(1)). \end{aligned}$$

Since by our choice of the sequence  $\{s_n, n \geq 1\}$

$$p_{n,d} = \left( \mathbb{P} \left( \bigcap_{i=1}^d \{M_{s_n}^{(i)} \leq u_n(x_i)\} \right) \right)^{t_n} + o(1) \quad \text{as } n \rightarrow \infty$$

we have

$$\begin{aligned} &\mathbb{P} \left( \bigcap_{i=1}^d \{M_{s_n}^{(i)} \leq u_n(x_i)\} \right) \\ &\geq \exp \left( -n \mathbb{P} \left( X_{n,1}^{(1)} > u_n(x_1), \bigcap_{t=1}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \right. \\ &\quad \left. - n \sum_{i=2}^d \mathbb{P} \left( X_{n,1}^{(i)} > u_n(x_i), \bigcap_{s=1}^{i-1} \{M_{r_n}^{(s)} \leq u_n(x_s)\}, \bigcap_{t=i}^d \{M_{1,r_n}^{(t)} \leq u_n(x_t)\} \right) \right) + o(1). \end{aligned}$$

Hence the theorem holds.  $\square$

**Acknowledgments.** Research of Hashorva and Weng was supported by the Swiss National Science Foundation grants 200021-134785, 200021-140633/1 and RARE -318984 (an FP7 Marie Curie IRSES Fellowship).

## REFERENCES

J.P. French and R.A. Davis (2013). The asymptotic distribution of the maxima of a Gaussian random field on a lattice. *Extremes*, **16**, 1–26.

- S. Engelke, Z. Kabluchko and M. Schlather (2014). Maxima of independent, non-identically distributed Gaussian vectors. *Bernoulli*, in press.
- B.G. Manjunath, M. Frick and R.-D. Reiss (2013). Some notes on extremal discriminant analysis. *J. Multiv. Analysis* **103**, 107–115.
- M. Frick and R.-D. Reiss (2013). Expansions and penultimate distributions of maxima of bivariate normal random vectors. *Statist. & Probab. Lett.* **83**, 2563–2568.
- E. Hashorva (2013). Minima and maxima of elliptical triangular arrays and spherical processes. *Bernoulli* **19**, 886–904.
- E. Hashorva, Z. Kabluchko and A. Wübker (2012). Extremes of independent chi-square random vectors. *Extremes* **15**, 35–42.
- E. Hashorva and Z. Weng (2013). Limit laws for extremes of dependent stationary Gaussian arrays. *Statist. & Probab. Lett.* **83**, 320–330.
- T. Hsing, J. Hüsler and R.-D. Reiss (1996). The extremes of a triangular array of normal random variables. *Ann. Appl. Probab.* **6**, 671–686.
- J. Hüsler and R.-D. Reiss (1989). Maxima of normal random vectors: between independence and complete dependence. *Statist. & Probab. Lett.* **7**, 283–286.
- W.V. Li and Q.M. Shao (2001). Gaussian processes: inequalities, small ball probabilities and applications. *Stochastic Processes: Theory and Applications* **19**, 533–597.
- A.J. McNeil, R. Frey and P. Embrechts (2005). Quantitative Risk Management: Concepts, Techniques, and Tools. *Princeton University Press*.
- G.L. O’Brien. (1987). Extreme values for stationary and Markov sequences. *Ann. Probab.* **15**, 281–291.
- V.I. Piterbarg (1996). Asymptotic Methods in the Theory of Gaussian Processes and Fields. *American Mathematical Society, Providence, RI*.

ENKELEJD HASHORVA, DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE,, UNIL-DORIGNY, 1015 LAUSANNE, SWITZERLAND

*E-mail address:* Enkelejd.Hashorva@unil.ch

LIANG PENG, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332-0160,

*E-mail address:* Peng@math.gatech.edu

ZHICHAO WENG, DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY, 1015 LAUSANNE, SWITZERLAND

*E-mail address:* zhichao.weng@unil.ch